Chapter 4

Rotating Stars

In this chapter the theory of gravitation in flat space-time of chapter I will be applied to rotating stars. Rotating neutron stars (pulsars) are numerically computed. All the results of this chapter are contained in the work of [Kus 88] where additional details can be found.

4.1 Field Equations

The line-element is again given by (1.1) with (2.2).The gravitational potentials are:

$$\begin{aligned}&g_{11}=\frac{1}{f},\, g_{22}=\frac{r^2}{g},\, g_{33}=\frac{r^2\sin^2\vartheta}{d},\, g_{44}=-\frac{1}{h},\\&g_{12}=g_{21}=ar,\; g_{34}=g_{43}=br\sin\vartheta,\, g_{ij}=0(else)\end{aligned} \tag{4.1}$$

Where the six functions f, g, d, h, a, b depend on r and ϑ. Put

$$\Omega_1=\frac{1}{fg}-a^2,\;\; \Omega_2=\frac{1}{dh}+b^2 \tag{4.2}$$

Then, we get

$$(-G)^{1/2}=r^2 sin\vartheta(\Omega_1\Omega_2)^{1/2}. \tag{4.3}$$

The energy-momentum tensor of matter is given by perfect fluid (1.28) where ρ, p and (u^i) are functions of r and ϑ. Let us assume that the star is rotating with constant angular velocity about an axis then the four-velocity is

$$\left(u^i\right)=\left(\frac{dr}{d\tau},\frac{d\vartheta}{d\tau},\frac{d\varphi}{d\tau},c\frac{dt}{d\tau}\right)=(0,0,\omega,c)\frac{dt}{d\tau}. \tag{4.4}$$

Put

$$\begin{aligned}&z_1=-\omega^2r^2\sin^2\vartheta\frac{1}{d}-cr\omega b\sin\vartheta\\&z_2=\frac{c^2}{h}-cr\omega b\sin\vartheta\end{aligned} \tag{4.5}$$

Then, we get from relation (1.8)

$$\frac{d\tau}{dt}=\frac{1}{c}(z_1+z_2)^{1/2}. \tag{4.6}$$

Hence, we receive from (4.4) the four-velocity

$$u^1=u^2=0, u^3=\omega c(z_1+z_2)^{-1/2}, u^4=c^2(z_1+z_2)^{-1/2}. \tag{4.7}$$

Therefore, the energy-momentum tensor of matter (1.28) has by virtue of (4.1) and (4.7) the form

$$
\begin{aligned}
T(M)^i_j &= pc^2 \ (i=j=1,2) \\
&= pc^2 - (\rho+p)c^2 z_1/(z_1+z_2) \ (i=j=3) \\
&= pc^2 - (\rho+p)c^2 z_2/(z_1+z_2) \ (i=j=4) \\
&= -(\rho+p)\omega c\, z_2/(z_1+z_2) \ (i=3, j=4) \\
&= -(\rho+p)(c^3/\omega)\, z_1/(z_1+z_2) \ (i=4, j=3) \\
&= 0. \ (else)
\end{aligned}
\tag{4.8}
$$

The formal representation of the energy-momentum tensor of the gravitational field with the aid of the potential functions f, g, d, h, a, b will now be stated. With the aid of complicated tensors D_{ij} (i, j=1,2) with $D_{12} = D_{21}$ which contain quadratic expressions of first order derivatives of the potentials let us define new tensors

$$
H^i_j = g^{ik} D_{jk}. \tag{4.9}
$$

Then, the components of the energy-momentum tensor of the gravitational field needed subsequently for the field equations are

$$
\begin{aligned}
T(G)^i_j &= \frac{(\Omega_1\Omega_2)^{1/2}}{16\kappa}(H^1_1 - H^2_2 - H^3_3) \quad \text{(i=j=1)} \\
&= \frac{(\Omega_1\Omega_2)^{1/2}}{16\kappa}(-H^1_1 + H^2_2 - H^3_3) \quad \text{(i=j=2)} \\
&= \frac{(\Omega_1\Omega_2)^{1/2}}{16\kappa}(-H^1_1 - H^2_2 + H^3_3) \quad \text{(i=j=3)} \\
&= -\frac{(\Omega_1\Omega_2)^{1/2}}{16\kappa}(H^1_1 - H^2_2 - H^3_3) \quad \text{(i=j=4)} \\
&= \frac{(\Omega_1\Omega_2)^{1/2}}{8\kappa} H^1_2 \quad \text{(i=1, j=2)}
\end{aligned}
\tag{4.10}
$$

To get the field equations we put for any function $\beta(r, \vartheta)$:

$$
\beta_{(1)} = \frac{\partial\beta}{\partial r}, \quad \beta_2 = \frac{\partial\beta}{\partial\vartheta}, \quad \beta_{12} = \frac{\partial^2\beta}{\partial r\partial\vartheta} = \frac{\partial^2\beta}{\partial\vartheta\partial r}. \tag{4.11}
$$

Furthermore, we define for $i = 1,2$

$$
\begin{aligned}
F_i &= \frac{1}{f}\left(\frac{1}{g\Omega_1}\right)_{(i)} - a\left(\frac{a}{\Omega_1}\right)_{(i)} + \frac{a}{\Omega_1}\left(\frac{1}{f}+\frac{1}{g}\right)\delta_{i2} \\
G_i &= \frac{1}{g}\left(\frac{1}{f\Omega_1}\right)_{(i)} - a\left(\frac{a}{\Omega_1}\right)_{(i)} - \frac{a}{\Omega_1}\left(\frac{1}{f}+\frac{1}{g}\right)\delta_{i2} \\
D_i &= \frac{1}{d}\left(\frac{1}{h\Omega_2}\right)_{(i)} + b\left(\frac{b}{\Omega_2}\right)_{(i)} \\
H_i &= \frac{1}{h}\left(\frac{1}{d\Omega_2}\right)_{i} + b\left(\frac{b}{\Omega_2}\right)_{(i)} \\
A_i &= a\left(\frac{1}{g\Omega_1}\right)_{(i)} - \frac{1}{g}\left(\frac{a}{\Omega_1}\right)_{(i)} + \left(-1+\frac{1+a^2g^2}{\Omega_1 g^2}\right)\delta_{i2} \\
B_i &= b\left(\frac{1}{h\Omega_2}\right)_{(i)} - \frac{1}{h}\left(\frac{b}{\Omega_2}\right)_{(i)} \\
M_i &= a\left(\left(\frac{1}{f\Omega_1}\right)_{(i)} + \left(\frac{1}{g\Omega_1}\right)_{(i)}\right) - \left(\frac{a}{\Omega_1}\right)_{(i)}\left(\frac{1}{f}+\frac{1}{g}\right) + \frac{1}{\Omega_1}\left(\frac{1}{g^2}-\frac{1}{f^2}\right)\delta_{i2} \\
N_i &= \frac{1}{f}\left(\frac{1}{g\Omega_1}\right)_{(i)} - \frac{1}{g}\left(\frac{1}{f\Omega_1}\right)_{(i)} + \frac{2a}{\Omega_1}\left(\frac{1}{f}+\frac{1}{g}\right)\delta_{i2}.
\end{aligned} \tag{4.12}
$$

In addition put

$$
\begin{aligned}
&y_1 = \frac{\sin^2\vartheta}{g} - a\sin\vartheta\cos\vartheta,\ y_2 = \frac{\sin^2\vartheta}{f} + a\sin\vartheta\cos\vartheta, \\
&y_3 = \frac{\cos^2\vartheta}{f} - a\sin\cos\vartheta,\ y_4 = \frac{\cos^2\vartheta}{g} + a\sin\vartheta\cos\vartheta, \\
&y_5 = -a\cos^2\vartheta + \frac{\sin\vartheta\cos\vartheta}{g},\ y_6 = a\sin^2\vartheta + \frac{\sin\vartheta\cos\vartheta}{g}
\end{aligned} \tag{4.13}
$$

The field equations (1.24) with $\Lambda = 0$ give the following system of differential equations:

$$
\begin{aligned}
&\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(\Omega_1\Omega_2)^{1/2}\left(\frac{1}{g\Omega_1}F_1 - \frac{a}{r\Omega_1}F_2\right)\right] \\
&+\frac{1}{r^2 sin\vartheta}\frac{\partial}{\partial\vartheta}\left[sin\vartheta(\Omega_1\Omega_2)^{1/2}\left(-\frac{ar}{\Omega_1}F_1 + \frac{1}{f\Omega_1}F_2\right)\right]
\end{aligned} \tag{4.14}
$$

$$-(\Omega_1\Omega_2)^{1/2}\left(-\frac{a}{r\Omega_1}M_1+\frac{1}{fr^2\Omega_1}M_2\right)$$

$$-\frac{(\Omega_1\Omega_2)^{1/2}}{r^2 hsin^2\vartheta\Omega_2}\left(\frac{y_1}{d\Omega_1}-\frac{y_2}{h\Omega_2}\right)$$

$$=2\kappa(\rho-p)c^2+\frac{1}{2}(\Omega_1\Omega_2)^{1/2}\left(-\frac{a}{r\Omega_1}D_{12}+\frac{1}{g\Omega_1}D_{11}\right),$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(\Omega_1\Omega_2)^{1/2}\left(\frac{1}{g\Omega_1}G_1-\frac{a}{r\Omega_1}G_2\right)\right]$$

$$+\frac{1}{r^2 sin\vartheta}\frac{\partial}{\partial\vartheta}\left[sin\vartheta(\Omega_1\Omega_2)^{1/2}\left(-\frac{ar}{\Omega_1}G_1+\frac{1}{f\Omega_1}G_2\right)\right]$$

$$+(\Omega_1\Omega_2)^{1/2}\left(-\frac{a}{r\Omega_1}M_1+\frac{1}{fr^2\Omega_1}M_2\right)$$

$$-\frac{(\Omega_1\Omega_2)^{1/2}}{r^2 hsin^2\vartheta\Omega_2}\left(\frac{y_3}{d\Omega_1}-\frac{y_4}{h\Omega_2}\right)$$

$$=2\kappa(\rho-p)c^2+\frac{1}{2}(\Omega_1\Omega_2)^{1/2}\left(-\frac{a}{r\Omega_1}D_{12}+\frac{1}{fr^2\Omega_1}D_{22}\right),$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(\Omega_1\Omega_2)^{1/2}\left(\frac{1}{g\Omega_1}D_1-\frac{a}{r\Omega_1}D_2\right)\right]$$

$$+\frac{1}{r^2 sin\vartheta}\frac{\partial}{\partial\vartheta}\left[sin\vartheta(\Omega_1\Omega_2)^{1/2}\left(-\frac{ar}{\Omega_1}D_1+\frac{1}{f\Omega_1}D_2\right)\right]$$

$$+\frac{(\Omega_1\Omega_2)^{1/2}}{r^2 hsin^2\vartheta\Omega_2}\left[\frac{1}{d\Omega_1}(y_1+y_3)-\frac{1}{h\Omega_2}(y_2+y_4)\right]$$

$$=2\kappa c^2\left[(\rho-p)-2(\rho+p)\frac{z_1}{z_1+z_2}\right]$$

$$+\frac{(\Omega_1\Omega_2)^{1/2}}{r^2 hsin^2\vartheta\Omega_2}\left[\frac{1}{d\Omega_1}(y_1+y_3)+\frac{1}{h\Omega_2}(y_2+y_4)-2\right],$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(\Omega_1\Omega_2)^{1/2}\left(\frac{1}{g\Omega_1}H_1-\frac{a}{r\Omega_1}H_2\right)\right]$$

$$+\frac{1}{r^2 sin\vartheta}\frac{\partial}{\partial\vartheta}\left[sin\vartheta(\Omega_1\Omega_2)^{1/2}\left(-\frac{ar}{\Omega_1}H_1+\frac{1}{f\Omega_1}H_2\right)\right]$$

$$=2\kappa c^2\left[(\rho-p)-2(\rho+p)\frac{z_2}{z_1+z_2}\right],$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(\Omega_1\Omega_2)^{1/2}\left(\frac{1}{g\Omega_1}A_1-\frac{a}{r\Omega_1}A_2\right)\right]$$

$$+\frac{1}{r^2 sin\vartheta}\frac{\partial}{\partial\vartheta}\left[sin\vartheta(\Omega_1\Omega_2)^{1/2}\left(-\frac{ar}{\Omega_1}A_1+\frac{1}{f\Omega_1}A_2\right)\right]$$

$$+(\Omega_1\Omega_2)^{1/2}\left(-\frac{a}{r\Omega_1}N_1+\frac{1}{r^2 f\Omega_1}N_2\right)$$

$$-\frac{(\Omega_1\Omega_2)^{1/2}}{r^2 h sin^2\vartheta\Omega_2}\left(\frac{y_5}{d\Omega_1}-\frac{y_6}{h\Omega_2}\right)$$

$$=\frac{1}{2r}(\Omega_1\Omega_2)^{1/2}\left(-\frac{a}{r\Omega_1}D_{22}+\frac{1}{g\Omega_1}D_{12}\right),$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(\Omega_1\Omega_2)^{1/2}\left(\frac{1}{g\Omega_1}B_1-\frac{a}{r\Omega_1}B_2\right)\right]$$

$$+\frac{1}{r^2 sin\vartheta}\frac{\partial}{\partial\vartheta}\left[sin\vartheta(\Omega_1\Omega_2)^{1/2}\left(-\frac{ar}{\Omega_1}B_1+\frac{1}{f\Omega_1}B_2\right)\right]$$

$$+\frac{(\Omega_1\Omega_2)^{1/2}}{r^2 h sin^2\vartheta\Omega_2}\frac{b}{\Omega_1}(y_1+y_3)$$

$$=-4\kappa c\omega r sin\vartheta(\rho+p)\frac{z_2}{z_1+z_2}.$$

Hence, we have six differential equations for the six potential functions f, g, d, h, a, b. It is worth mentioning that the equation (4.14) for the function a does not depend on the matter tensor (4.8) but it cannot be omitted because the equation implies that $a \neq 0$.

4.2 Equations of Motion

We get from the equations (1.29a) by the use of (4.1) and (4.8) the following two differential equations

$$\begin{aligned}\frac{dp}{dr}=&-\frac{1}{2}\left[\frac{1}{fg\Omega_1}\left(\frac{f_{(1)}}{f}+\frac{g_{(1)}}{g}\right)+\frac{1}{dh\Omega_2}\left(\frac{d_{(1)}}{d}+\frac{h_{(1)}}{h}\right)\right]p\\&+\frac{1}{r}\left[\frac{b(rb)_{(1)}}{\Omega_2}-\frac{a(ra)_1}{\Omega_1}\right]p+\frac{1}{r}\left[\frac{1}{fg\Omega_1}+\frac{1}{dh\Omega_2}-2\right]p\\&-\frac{1}{2}\frac{(z_1+z_2)_{(1)}}{z_1+z_2}(\rho+p),\\\frac{dp}{d\vartheta}=&-\frac{1}{2}\left[\frac{1}{fg\Omega_1}\left(\frac{f_{(2)}}{f}+\frac{g_{(2)}}{g}\right)+\frac{1}{dh\Omega_2}\left(\frac{d_{(2)}}{d}+\frac{h_{(2)}}{h}\right)-2\left(\frac{bb_{(2)}}{\Omega_2}-\frac{aa_{(2)}}{\Omega_1}\right)\right]p\\&-\frac{1}{2}\frac{(z_1+z_2)_{(2)}}{z_1+z_2}(\rho+p).\end{aligned}\tag{4.15}$$

The conservation of the mass (1.29b) is fulfilled by virtue of (4.4).

The boundary of the rotating star is given by the condition

$$\mathrm{p}(r,\vartheta)=0 \tag{4.16}$$

implying that the boundary $r(\vartheta)$ depends on the angle ϑ by virtue of the rotation of the star. Hence, spherical symmetry cannot hold.

It is worth mentioning that we have two equations of motion (4.15) but by virtue of the equations of state of the form (2.13) we have only one function $\rho(r,\vartheta)$. This is connected with the assumption that we consider a rigid body rotating about a fixed axis with constant angular velocity $\frac{d\varphi}{dt}=\omega$. Hence, a perfect fluid for a rigid body rotating about a fixed axis with constant angular velocity ω and a velocity of the form (4.4) does not exist because gravitational forces work. This gives a solution to the paradox of a uniformly rotating disc noted by Ehrenfest and considered as justification for the introduction of non-Euclidean geometry in general relativity theory of Einstein (see e.g.[Mol 72]). It should be mentioned that any transformation of the pseudo-Euclidean geometry conserves the flat space-time metric. Therefore, we have for a body rotating about an axis to introduce additional velocities $\frac{dr}{dt}\neq 0$ and perhaps also $\frac{d\vartheta}{dt}\neq 0$ as functions of r and ϑ which will give new differential equations. A simpler possibility without great changes of the equations is the assumption that ω is a function of r and ϑ.

Rotating stars studied with the aid of general relativity can be found e.g. in the article [Dem 85].

4.3 Rotating Neutron Stars

In the following, we consider a rotating neutron star with constant angular velocity ω approximately described by the equations (4.14) and (4.15). The results of this sub-chapter can be found in the work of [Kus 88].

To simplify the equations (4.14) and (4.15) all small functions a and b in the non-linear expressions are neglected, i.e. we consider to the lowest order a non-rotating star studied in chapter 2.10 where all functions only depend on r . We put for any function

$$\beta(r,\vartheta) \approx \beta_0(r) + \beta_\varepsilon(r,\vartheta). \tag{4.17}$$

The function $\beta_0(r)$ describes the non-rotating star and β_ε is the small correction implied by the rotation. Then, we have by virtue of sub-chapter 2.10

$$d_0 = g_0, \quad a_0 = b_0 = \omega_0 = 0. \tag{4.18}$$

For the angular velocity it is assumed that the expression

$$\Omega := \frac{K}{c}\omega << 1. \tag{4.19}$$

The transformations

$$p = \left(\frac{-G}{-\eta}\right)^{1/2} \tilde{p}\ , \rho = \left(\frac{-G}{-\eta}\right)^{1/2} \tilde{\rho} \tag{4.20}$$

give new expressions $\tilde{p}$ and $\tilde{\rho}$ for pressure and density. In the following quadratic expressions of a and b are omitted. Then, these approximations imply with the aid of (4.3) and (4.2)

$$p = \frac{\tilde{p}}{(fgdh)^{1/2}}, \quad \rho = \frac{\tilde{\rho}}{(fgdh)^{1/2}}. \tag{4.21}$$

Now, the equations of motion (4.15) can for $i = 1,2$ approximately be written by the use of (4.4) and by neglecting quadratic expressions of the small quantities a and b in the form

$$\tilde{p}_{(i)} = -\frac{1}{2}(\tilde{\rho} + \tilde{p})\frac{(z_1+z_2)_{(i)}}{z_1+z_2}. \tag{4.22}$$

Let us now assume an equation of state

$$\tilde{p} = \tilde{p}(\tilde{\rho}) \tag{4.23a}$$

Therefore, on the above approximations and the assumption (4.23a) with (4.20) the two differential equations (4.22) depend on one another in contrast to the general case (4.15). In the following, we use an equation of state

$$\tilde{p} = C\tilde{\rho}^{1+\gamma}, \gamma = 2/3 \tag{4.23b}$$

with a suitable constant C. Then, the two differential equations (4.22) have a unique solution for $\tilde{\rho}$ which will not be given. By virtue of (4.23b) $\tilde{p}$ can be calculated and we get p and ρ by (4.21). It seems more natural to assume an equation of state for p as function of ρ instead of $\tilde{p}$ as function of $\tilde{\rho}$. Hence, we see that a rigid body can be approximated on some assumptions but it does not really exist. With the approximation of the form (4.17) for the pressure $\tilde{p}$ and $\tilde{\rho}$ we can calculate the pressure p_0 and the density ρ_0 of the non-rotating neutron star and the corresponding approximated values p_ε and ρ_ε which follow from the above equation. Substituting all the expressions into the field equations (4.14) we get to the lowest order three ordinary differential equations for the functions $f_0(r)$, $g_0(r)$ and $h_0(r)$ describing the non-rotating neutron star. In addition, by linearization of the equations (4.14) we get six linear partial differential equations for the approximated functions f_ε, g_ϵ, d_ε, h_ε, a_ε and b_ε depending on r and ϑ. It is worth mentioning that the equations describing the non-rotating neutron star are different from those of sub-chapter 2.10 by virtue of the different equation of state.

Numerical methods are used for the solution of the problem where angular velocities $\omega \in [570, 1034]\frac{1}{sec}$ are considered fulfilling the assumption (4.19) and being observed (see $[Bac\ 82]$). We will give some of these results in the following table where in the column with $\omega = 0$ the results for the non-rotating star are stated. Here, a and b denote the semi-major and semi-minor axes of the flattened rotating body (pulsar) where $a = b$ gives the neutron star.

Table

$\tilde{\rho}(0)\cdot 10^{-15}$ g/cm^3	$\tilde{p}(0)\cdot 10^{-15}$ g/cm^3	$M/M_\odot$	a km	b km	ω $1/sec$
0.895	0.0850	1.30	14.4	14.4	0
0.895	0.0850	1.31	14.4	14.4	570
0.895	0.0850	1.33	14.4	14.3	1000
0.898	0.0852	1.35	14.6	14.0	2000
0.904	0.0855	1.38	14.6	13.8	3000
0.916	0.0862	1.42	14.91	13.3	4030
1.53	0.317	1.98	14.6	14.6	0
1.54	0.138	2.06	14.8	14.2	1000
2.01	0.547	2.09	14.0	14.0	0
2.03	0.550	2.17	14.2	13.6	1000

The table shows that greater angular velocities and greater densities with greater pressure give greater deviations of the mass and semi-axes. For too small angular velocities and densities there are negligible deviations of mass and of semi-axes from those of non-rotating neutron stars.

Furthermore, it follows by comparison with the results of sub-chapter 2.10 that an equation of state of the form (2.13) gives a greater mass than an equation of state (4.23a) with (4.21).

More details of the approximations and the numerical computations of rotating stars can be found in the work of [Kus 88].

Results about neutron stars based on the theory of general relativity are given by [Har 67] and can also be found in [Dem 85] containing further references.