# Chapter 6

## Post-Newtonian of Spherical Symmetry

In this chapter spherically symmetric stars with their gravitational fields are studied to post-Newtonian approximation. The equations of motion of the star and the energy-momentum tensor are given. All these results agree with the corresponding results of general relativity to 1-post-Newtonian accuracy whereas to 2-post-Newtonian approximation the results are different from one another. In particular, the theorem of Birkhoff is not valid. Hence, the theory of gravitation in flat space-time and the general theory of relativity are different to this accuracy.

## 6.1 Post-Newtonian Approximation of Non-Stationary Stars

The equations describing a non-stationary spherically symmetric star depending on the distance from the centre $r$ of the star and the time $\tilde{t}$ are given in chapter III. The field equations are stated in formula (3.12) with $\rho \rightarrow \rho\left(1+\frac{\Pi}{c^2}\right)$ where $\Pi$ denotes the specific internal energy. The equations of motion are stated by (3.13) and the conservation of the whole energy-momentum is given by (3.14). Furthermore, we have an equation of state (3.15).It is worth to mention that the equations of field (3.12) together with the equations of motion (3.13) imply the equations of the whole energy-momentum (3.14). Hence, the relations (3.14) can be omitted. The conservation law of mass (1.29b) has the form

$$\frac{1}{r^2 F_{(4)}}\left(\frac{\partial}{\partial r}\left(r^2 F_{(4)} \rho u^1\right)+\frac{\partial}{\partial c\tilde{t}}\left(r^2 F_{(4)} \rho u^4\right)\right)=0 \tag{6.1}$$

where $u^4$ is given by (3.6). Hence, we have eight functions $f, g, h, F, \rho, p, \Pi$ and $u^1$ depending on $r$ and $\tilde{t}$ and eight independent equations: (3.12) (four equations), (3.13) (two equations), (6.1) (one equation) and (3.15) (one equation).

A suitable combination of the equations of motion (3.13) yields

$$\begin{aligned}
&\frac{\partial \Pi}{\partial \tilde{t}}-\frac{pc^2}{\rho^2}\frac{\partial \rho}{\partial \tilde{t}}-\frac{pc^2}{\rho^2}\frac{\partial}{\partial \tilde{t}}\log(fg^2F_{(4)}^2 h)^{1/2} \\
&+c\frac{u^1}{u^4}\left\{\frac{\partial \Pi}{\partial r}-\frac{pc^2}{\rho^2}\frac{\partial \rho}{\partial r}-\frac{pc^2}{\rho^2}\frac{\partial}{\partial r}\log(fg^2F_{(4)}^2 h)^{1/2}\right\} \\
&=0
\end{aligned} \tag{6.2}$$

Hence, we replace equation (3.13b) by the simpler equation (6.2). In the following we introduce the radial velocity $v(r,\tilde{t})$ instead of $u^1$ and $u^4$ given by

$$u^1 = v\frac{d\tilde{t}}{d\tau} = v\left(\frac{h}{1-\frac{h}{f}\left|\frac{v}{c}\right|^2}\right)^{1/2}, \; u^4 = c\frac{d\tilde{t}}{d\tau} = c\left(\frac{h}{1-\frac{h}{f}\left|\frac{v}{c}\right|^2}\right)^{1/2}. \tag{6.3}$$

The post-Newtonian approximation assumes

$$\begin{gathered}\rho = 0(1),\, v = 0(1),\, \frac{p}{\rho} = 0\left(\frac{1}{c^2}\right), \\ \frac{\Pi}{\rho} = 0\left(\frac{1}{c^2}\right),\, \frac{\partial}{\partial r} = 0(1),\, \frac{\partial}{\partial c\tilde{t}} = 0\left(\frac{1}{c}\right).\end{gathered} \tag{6.4}$$

We make the following ansatz for the post-Newtonian approximation

$$\begin{gathered}f = 1 - \frac{2}{c^2}U_1 + \frac{1}{c^4}S_1 = 0\left(\frac{1}{c^4}\right),\, g = 1 - \frac{2}{c^2}U_2 + \frac{1}{c^4}S_2 = 0\left(\frac{1}{c^4}\right) \\ F_{(4)}^2 h = 1 + \frac{2}{c^2}U_3 + \frac{1}{c^4}S_3 = 0\left(\frac{1}{c^4}\right),\, F = c\tilde{t} + \frac{1}{c^3}S_4 = 0\left(\frac{1}{c^3}\right)\end{gathered} \tag{6.5}$$

Here, the functions $U_i$ (i=1,2,3), $S_i$(i=1,2,3) are of order $O(1)$ and depend on $r$ and $\tilde{t}$.

The boundary conditions must converge to zero as $r$ goes to infinity. It holds

$$F_{(4)} = 1 + \frac{1}{c^4}\frac{\partial S_4}{\partial \tilde{t}} = O\left(\frac{1}{c^4}\right) \tag{6.6a}$$

and

$$h = 1 + \frac{2}{c^2}U_3 + \frac{1}{c^4}\left(S_3 - 2\frac{\partial S_4}{\partial \tilde{t}}\right) = O\left(\frac{1}{c^4}\right). \tag{6.6b}$$

The post-Newtonian approximation implies

$$S_1 = S_2 = 0. \tag{6.7}$$

Hence, we get from the field equations (3.12) to $O\left(\frac{1}{c^2}\right)$ that

$$U_1 = U_2 = U_3 = U \tag{6.8}$$

which satisfies the differential equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) = -4\pi k\rho. \tag{6.9a}$$

The third field equation gives to $O\left(\frac{1}{c^4}\right)$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial(S_3-2U^2)}{\partial r}\right) = 2\frac{\partial^2 U}{\partial \tilde{t}^2} - 8\pi k\rho\left(\Pi + 3\frac{pc^2}{\rho} + 2v^2\right) \tag{6.9b}$$

where $\frac{\partial^2 U}{\partial \tilde{t}^2}$ must be calculated to $O(1)$. The last field equation implies to $O\left(\frac{1}{c^3}\right)$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial^2 S_4}{\partial r^2}\right) - \frac{2}{r^2}\frac{\partial S_4}{\partial r} = 16\pi k\rho v.$$

This equation can be integrated by the use of the boundary conditions implying

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial S_4}{\partial r}\right) = 16\pi k\int_{-\infty}^{r}\rho v dx. \tag{6.9c}$$

The differential equations (6.9) must be solved by the use of the boundary conditions.

The solutions of (6.9a) and (6.9c) are

$$U = 4\pi k\left\{\frac{1}{r}\int_0^r x^2\,\rho(x,\tilde{t})dx - \int_\infty^r x\rho(x,\tilde{t})dx\right\}, \tag{6.10a}$$

$$S_4 = \frac{16\pi k}{3}\left\{\frac{1}{r}\int_0^r x^3\rho v dx + \frac{1}{2}r^2\int_\infty^r \rho v dx - \frac{3}{2}\int_\infty^r x^2\rho v dx\right\}. \tag{6.10b}$$

Equation (6.1) gives by the use of (6.3), (6.5) and (6.6a) to $O(1)$

$$\frac{\partial\rho}{\partial\tilde{t}} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho v) = 0.$$

Differentiation of equation (6.9a) gives by the use of this relation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{\partial U}{\partial\tilde{t}}\right)\right) = -4\pi k\frac{\partial\rho}{\partial\tilde{t}} = 4\pi k\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho v).$$

Elementary integration yields to $O(1)$

$$\frac{\partial U}{\partial\tilde{t}} = 4\pi k\int_\infty^r \rho v dx. \tag{6.11}$$

Equation (6.9c) gives by differentiation and the use of (6.11)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{\partial S_4}{\partial\tilde{t}}\right)\right) = 16\pi k\frac{\partial}{\partial\tilde{t}}\int_\infty^r \rho v dx = 4\frac{\partial^2 U}{\partial\tilde{t}^2}.$$

Therefore, equation (6.9b) can be rewritten in the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(S_3-2U^2-\frac{1}{2}\frac{\partial S_4}{\partial\tilde{t}}\right)\right)=-8\pi k\rho\left(\Pi+3\frac{pc^2}{\rho}+2v^2\right)\quad(6.12)$$

Let us now introduce the potentials in analogy to (5.14)

$$\begin{aligned}\phi_1&=4\pi k\left(\frac{1}{r}\int_0^r x^2\rho v^2dx-\int_\infty^r x\rho v^2dx\right),\\ \phi_3&=4\pi k\left(\frac{1}{r}\int_0^r x^2\rho\Pi dx-\int_\infty^r x\rho\Pi dx\right),\\ \phi_4&=4\pi k\left(\frac{1}{r}\int_0^r x^2pc^2dx-\int_\infty^r xpc^2dx\right).\end{aligned}\quad(6.13)$$

The differential equation (6.12) has the solution

$$S_3=2U^2+\frac{1}{2}\frac{\partial S_4}{\partial\tilde{t}}+2(2\phi_1+\phi_3+3\phi_4).\quad(6.10c)$$

The relations (6.10) give together with (6.5), (6.6), (6.7) and (6.8) the post-Newtonian approximation.

The energy-momentum tensor of matter (3.7) can now be given to accuracy

$$\begin{aligned}T(M)^i_j&=0\left(\frac{1}{c^2}\right),(i=j=1-4)\\ &=0\left(\frac{1}{c^3}\right),(i=1;\,j=4)\\ &=0\left(\frac{1}{c}\right),(i=4;\,j=1)\end{aligned}\quad(6.14a)$$

and the corresponding tensor of the gravitational field

$$\begin{aligned}T(G)^i_j&=0\left(\frac{1}{c^2}\right),(i=j=1,2,3,4)\\ &=0\left(\frac{1}{c^3}\right),(i=1;\,j=4),(i=4;\,j=1)\end{aligned}\quad(6.14b)$$

The expressions of these tensors to post-Newtonian approximation are omitted but they can be found in the article [$Pet\ 94a$].

We will now give the equations of motion to post-Newtonian accuracy of $O\left(\frac{1}{c^2}\right)$. We use the differential equations (3.13) with matter the tensor (3.7), the differential equation (6.2) and the conservation law of mass (6.1). The post-Newtonian approximations (6.5) with (6.6), (6.7) and (6.8) are used. Furthermore, the representations (6.10) are introduced. After longer calculations the following post-Newtonian approximations to $O\left(\frac{1}{c^2}\right)$ are received:

$$\begin{aligned}\frac{\partial\rho}{\partial\tilde{t}} &= -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho v)\\ &+\frac{1}{c^2}\left[v\frac{\partial pc^2}{\partial r}+4\pi k\left(2\frac{v}{r^2}\int_0^r x^2\rho dx+\int_\infty^r v\rho dx\right)\right]\end{aligned}\tag{6.15a}$$

$$\begin{aligned}\frac{\partial\Pi}{\partial\tilde{t}} &= -v\frac{\partial\Pi}{\partial r}\\ &+\frac{pc^2}{\rho^2}\left[-\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v)+\frac{1}{c^2}\left(v\frac{1}{\rho}\frac{\partial pc^2}{\partial r}+4\pi k\left(4\frac{v}{r^2}\int_0^r x^2\rho dx+3\int_r^\infty v\rho dx\right)\right)\right]\end{aligned}\tag{6.15b}$$

$$\begin{aligned}\frac{\partial v}{\partial\tilde{t}} &= -v\frac{\partial v}{\partial r}-\frac{1}{\rho}\frac{\partial pc^2}{\partial r}\left[1-\frac{1}{c^2}\left(\Pi+\frac{pc^2}{\rho}+4U+2v^2\right)\right]\\ &+\frac{\partial U}{\partial r}\left(1+\frac{2}{c^2}\left(\frac{pc^2}{\rho}-2U\right)\right)\\ &-\frac{4\pi k}{c^2}r\int_r^\infty\frac{1}{x}\rho\left(2v^2+\frac{4\pi k}{x}\int_0^x y^2\rho(y,\tilde{t})dy\right)dx\\ &-\frac{4\pi k}{c^2}\frac{1}{r^2}\int_0^r x^2\rho\left(v^2+\Pi+\frac{4\pi k}{x}\int_0^x y^2\rho(y,\tilde{t})dy\right)dx\\ &+\frac{v}{c^2}\frac{1}{r^2}\frac{\partial(r^2 v)}{\partial r}\left(\frac{pc^2}{\rho}+\frac{pc^2}{\rho}\frac{\partial}{\partial\Pi}\left(\frac{pc^2}{\rho}\right)+\rho\frac{\partial}{\partial\rho}\left(\frac{pc^2}{\rho}\right)\right)\\ &+\frac{12\pi k}{c^2}v\left(\frac{v}{r^2}\int_0^r x^2\rho dx+\int_r^\infty\rho v dx\right).\end{aligned}\tag{6.15c}$$

In the equation (6.15c) we have to eliminate $U$ by relation (6.10a). Then, the equations (6.15) are three integro-differential equations to post-Newtonian approximation $O\left(\frac{1}{c^2}\right)$ for the three unknown functions $v, \rho$ and $\Pi$. Let us assume an equation of state

$$\frac{p}{\rho}=\Pi\, B(\Pi,\rho)\tag{6.16}$$

with a suitable function $B$. The boundary $R(\tilde{t})$ of the star follows from (6.15b) with $r = R(\tilde{t})$:

$$\frac{d}{d\tilde{t}}\left(\Pi(R(\tilde{t}),\tilde{t})\right) = \left(\frac{\partial\Pi(r,\tilde{t})}{\partial\tilde{t}} + v\frac{\partial\Pi(r,\tilde{t})}{\partial r}\right)(r = R(\tilde{t}))\sim p\left(R(\tilde{t})\right) = 0.$$

Therefore, we have that

$$\Pi(R(\tilde{t}),\tilde{t}) = \Pi(R(\tilde{t}_0),\tilde{t}_0) = 0 \tag{6.17}$$

if for a fixed time $\tilde{t}_0$ the relation $\Pi(R(\tilde{t}_0),\tilde{t}_0) = 0$ holds. Then, relation (6.17) defines the boundary of the non-stationary star to post-Newtonian accuracy. The equation (6.17) is independent of the equation of state (6.16) but (6.16) is in agreement with (6.17).

The detailed longer derivations of the equations (6.15) are given in $[Pet\ 94a]$.

We will now study the potentials in the exterior of the star, i.e. $r > R(\tilde{t})$. It follows from relation (6.5) with (6.6), (6.7), (6.8) and (6.10)

$$f \approx g \approx 1 - \frac{2}{c^2}U \tag{6.18a}$$

$$h \approx 1 + \frac{2}{c^2}U + \frac{1}{c^4}\left(2U^2 - \frac{3}{2}\frac{\partial S_4}{\partial\tilde{t}} + 2(2\phi_1 + \phi_3 + 3\phi_4)\right) \tag{6.18b}$$

with

$$U = \frac{4\pi k}{r}\int_0^\infty x^2\rho dx$$

$$\frac{\partial S_4}{\partial\tilde{t}} = \frac{16\pi k}{3}\frac{1}{r}\int_0^\infty x^3\frac{\partial(\rho v)}{\partial\tilde{t}}dx$$

$$2\phi_1 + \phi_3 + 3\phi_4 = \frac{4\pi k}{r}\int_0^\infty x^2\rho\left(2v^2 + \Pi + 3\frac{pc^2}{\rho}\right)dx.$$

It holds (see $[Pet\ 94a]$) that the gravitational mass to $O\left(\frac{1}{c^2}\right)$ is

$$M_g = 4\pi\int_0^\infty x^2\rho\left\{1 + \frac{1}{c^2}\left(\Pi + \frac{1}{2}U + v^2\right)\right\}dx \tag{6.19a}$$

Hence, relation (6.18a) gives to $O\left(\frac{1}{c^2}\right)$

$$f \approx g \approx 1 - 2\frac{kM_g}{c^2 r} \tag{6.19b}$$

Relation (6.18b) can be rewritten by the use of (6.19a) to $O\left(\frac{1}{c^4}\right)$

$$h \approx 1 + 2\frac{kM_g}{c^2 r} + \frac{1}{c^4}\left\{2\left(\frac{kM_g}{c^2 r}\right)^2 - \frac{8\pi k}{r}\int_0^\infty \left(x^3 \frac{\partial(\rho v)}{\partial \tilde{t}} - x^2 \rho\left(v^2 + 3\frac{pc^2}{\rho} - \frac{1}{2}U\right)\right)dx\right\}.$$

In the article $[Pet\ 94a]$ it is shown that the last expression vanishes. Therefore, we get to $O\left(\frac{1}{c^4}\right)$

$$h \approx 1 + 2\frac{kM_g}{c^2 r} + 2\left(\frac{kM_g}{c^2 r}\right)^2. \tag{6.19c}$$

The relations (6.19) show that in the exterior of the star the theorem of Birkhoff holds to post-Newtonian accuracy.

## 6.2 2-Post-Newtonian Approximation of a Non-Stationary Star

We will in this sub-chapter only give some results of 2-post-Newtonian approximation. The study is given in $[Pet\ 94b]$ where the results are derived. We make the ansatz

$$\begin{aligned} &f = 1 - \frac{2}{c^2}U + \frac{1}{c^4}S_1, g = 1 - \frac{2}{c^2}U + \frac{1}{c^4}S_2, \\ &F_{(4)}^2 h = 1 + \frac{2}{c^2}U + \frac{1}{c^4}S_3 + \frac{1}{c^6}S_5, F = c\tilde{t} + \frac{1}{c^3}S_4 + \frac{1}{c^5}S_6 \end{aligned} \tag{6.20}$$

where $S_1$, $S_2$, $S_5$ and $S_6$ are of order $O(1)$ and $U$, $S_3$ and $S_4$ are already given in chapter 6.1.

For 2-post-Newtonian the time-derivatives must be considered to higher approximations, i.e. let $y(r, \tilde{t})$ be any function then the following approximation is used

$$\frac{\partial y}{\partial \tilde{t}} = \left(\frac{\partial y}{\partial \tilde{t}}\right)_0 + \frac{1}{c^2}\left(\frac{\partial y}{\partial \tilde{t}}\right)_2 + \frac{1}{c^4}\left(\frac{\partial y}{\partial \tilde{t}}\right)_4 \tag{6.21}$$

where $\left(\frac{\partial y}{\partial \tilde{t}}\right)_0$ is the Newtonian approximation, $\left(\frac{\partial y}{\partial \tilde{t}}\right)_2 = O\ (1)$ and $\left(\frac{\partial y}{\partial \tilde{t}}\right)_4 = O\ (1)$ are the 1-post-Newtonian and the 2-post-Newtonian approximations. We get from (6.20) up to 2-post-Newtonianaccuracy

$$F_{(4)} = \frac{\partial F}{\partial c\tilde{t}} = 1 + \frac{1}{c^4}\left(\frac{\partial S_4}{\partial \tilde{t}}\right)_0 + \frac{1}{c^6}\left\{\left(\frac{\partial S_4}{\partial \tilde{t}}\right)_2 + \left(\frac{\partial S_6}{\partial \tilde{t}}\right)_0\right\} \tag{6.22a}$$

$$h = 1 + \frac{2}{c^2}U = \frac{1}{c^4}\left\{S_3 - 2\left(\frac{\partial S_4}{\partial \tilde{t}}\right)_0\right\} + \frac{1}{c^6}\left\{S_5 - 4U\left(\frac{\partial S_4}{\partial \tilde{t}}\right)_0 - 2\left(\frac{\partial S_4}{\partial \tilde{t}}\right)_2 - 2\left(\frac{\partial S_6}{\partial \tilde{t}}\right)_0\right\}. \tag{6.22b}$$

In analogy to chapter 6.1 the expressions (6.20) and (6.22) are substituted into the differential equations (3.12). We get by elementary longer calculations differential equations for the 2-post-Newtonian approximations $S_1$, $S_2$, $S_5$ and $S_6$ whereas the functions $U$, $S_4$ and $S_3$ are given by (6.10). The solutions of these equations are given as functions of $\rho$, $p$, $\Pi$ and $v$. It is worth to mention that $S_6$ implies divergent integrals by the standard 2-post Newtonian approximation. Hence, it is necessary to use retarded functions. Therefore, the expression of the energy tensor contains retardations implying gravitational waves of the order $O\left(\frac{1}{c^5}\right)$ . This is a well-known fact of higher order post-Newtonian approximations also by the use of general relativity theory. This may be the reason why higher order post-Newtonian approximations are not possible implying divergent integrals. Furthermore, the expressions for $S_6$, $\frac{\partial S_6}{\partial r}$ and $\frac{\partial S_6}{\partial \tilde{t}}$ are not of order $O(1)$. Therefore, they do not fulfil the condition on 2-post-Newtonian approximation. The whole energy-momentum tensors of matter and of gravitational field can be given. The equations of motion and the conservation of mass are also stated where the gravitational mass $M_g$ can be given to accuracy $O\left(\frac{1}{c^6}\right)$. We will now state the solution of the non-stationary star in the exterior, i.e. $r > R(\tilde{t})$.

It follows

$$\begin{aligned}
f &= 1 - 2\frac{kM_g}{c^2 r} + 2\left(\frac{kM_g}{c^2 r}\right)^2 \\
&\quad + \frac{1}{c^4}\left\{\frac{16}{15}\frac{(4\pi k)^2}{r^3}\int_0^\infty x^3\rho\int_0^x y^2\,\rho dy dx - \frac{8}{15}\frac{4\pi k}{r^3}\int_0^\infty x^4\rho v^2 dx\right\}, \\
g &= 1 - 2\frac{kM_g}{c^2 r} + 3\left(\frac{kM_g}{c^2 r}\right)^2 \\
&\quad - \frac{1}{c^4}\left\{\frac{8}{15}\frac{(4\pi k)^2}{r^3}\int_0^\infty x^3\rho\int_0^x y^2\rho dy dx - \frac{4}{15}\frac{4\pi k}{r^3}\int_0^\infty x^4\rho v^2\, dx\right\}, \\
h &= 1 + 2\frac{kM_g}{c^2 r} + 2\left(\frac{kM_g}{c^2 r}\right)^2 + 2\left(\frac{kM_g}{c^2 r}\right)^3
\end{aligned} \tag{6.23}$$

$$+\frac{1}{c^6}\left\{-\frac{8}{15}\frac{kM_g}{r}\frac{(4\pi k)^2}{r^3}\int_0^\infty x^3\rho\int_0^x y^2\rho dydx+\frac{4}{15}\frac{kM_g}{r}\frac{4\pi k}{r^3}\int_0^\infty x^4\,\rho v^2 dx\right\}$$

$$+\frac{1}{c^6}\left\{\frac{8}{9}\frac{(4\pi k)^2}{r^4}\left(\int_0^\infty x^3\,\rho v dx\right)^2-\frac{8}{9}\frac{kM_g}{r}\frac{4\pi k}{r}\left(\frac{\partial}{\partial\tilde{t}}\int_0^\infty x^3\,\rho v dx\right)_0\right\}$$

$$-\frac{1}{c^6}\frac{8}{15}\frac{4\pi k}{r}\left(\frac{\partial}{\partial\tilde{t}}\int_0^\infty x^3\rho\left(4\frac{pc^2}{\rho}v+v^3+x\frac{pc^2}{\rho}\frac{\partial v}{\partial x}\right)dx\right)_0$$

$$+\frac{1}{c^6}\left\{\frac{(4\pi k)^2}{r}\left(\frac{\partial}{\partial\tilde{t}}\left[\frac{32}{9}\int_0^\infty x^4\rho\int_\infty^x\rho v dydx+\frac{176}{15}\int_0^\infty x^2\rho v\int_0^x y^2\rho dydx\right]\right)_0\right\}$$

$$+\frac{1}{c^6}\frac{(4\pi k)^2}{r}\left(\frac{\partial}{\partial\tilde{t}}\left(\frac{64}{9}\int_0^\infty x\rho\int_0^x y^3\,\rho v dydx\right)\right)_0.$$

The derivations of all the mentioned results of chapter 6.2 are longer calculations and they are not trivial. Therefore, only the exterior potentials (6.23) of the star are stated. It immediately follows from (6.23) that Birkhoff's theorem is not valid by the use of 2-post-Newtonian approximation. This is in contrast to the theory of Einstein. Hence, flat space-time theory of gravitation and the general relativity theory of gravitation give different results to higher order approximations.

The equations (6.23) give for a static spherically symmetric star up to 2-post-Newtonian approximation

$$\begin{aligned}
f&=1-2\frac{kM_g}{c^2r}+2\left(\frac{kM_g}{c^2r}\right)^2+\frac{1}{c^4}\frac{16}{15}\frac{(4\pi k)^2}{r^3}\int_0^\infty x^3\rho\int_0^x y^2\rho dydx,\\
g&=1-2\frac{kM_g}{c^2r}+3\left(\frac{kM_g}{c^2r}\right)^2-\frac{1}{c^4}\frac{8}{15}\frac{(4\pi k)^2}{r^3}\int_0^\infty x^3\rho\int_0^x y^2\rho dydx,\\
h&=1+2\frac{kM_g}{c^2r}+2\left(\frac{kM_g}{c^2r}\right)^2+2\left(\frac{kM_g}{c^2r}\right)^3\\
&\quad-\frac{1}{c^6}\frac{8}{15}\frac{kM_g}{r}\frac{(4\pi k)^2}{r^3}\int_0^\infty x^3\rho\int_0^x y^2\rho dydx
\end{aligned}\tag{6.24}$$

It follows by comparing the two solutions (6.14) and (2.39) that the constant $A$ of (2.39) is of order $O(c^2)$ with

$$A=\frac{8}{15}\frac{(4\pi c)^2}{kM_g^3}\int_0^\infty x^3\rho\int_0^x y^2\,\rho dydx\,.\tag{6.25}$$

It is worth mentioning that the factors of the expressions $\left(\frac{K}{r}\right)^3$ in the formulae (2.39a) and (2.39b) are of order $O(1)$ and the factor of the expression $\left(\frac{K}{r}\right)^4$ in (2.39c) is of orderof $O(1), too$.But by virtue of (6.25) in the formulae (6.24) these factors are too great and do not satisfy 2-post-Newtonian approximation. Therefore, the exterior solution of a static spherically symmetric star is approximately given by

$$\begin{aligned} f &= 1 - 2\frac{kM_g}{c^2 r} + 2\left(\frac{kM_g}{c^2 r}\right)^2 + 2A\left(\frac{kM_g}{c^2 r}\right)^3, \\ g &= 1 - 2\frac{kM_g}{c^2 r} + 3\left(\frac{kM_g}{c^2 r}\right)^2 - A\left(\frac{kM_g}{c^2 r}\right)^3, \\ h &= 1 + 2\frac{kM_g}{c^2 r} + 2\left(\frac{kM_g}{c^2 r}\right)^2 + 2\left(\frac{kM_g}{c^2 r}\right)^3 - A\left(\frac{kM_g}{c^2 r}\right)^4 \end{aligned} \tag{6.26}$$

where $A = (c^2)$ is stated by formula (6.25). Estimates of $A$ fulfil the condition

$$A \gg 1$$

which is in agreement of (2.39) with (6.26).

All these results with detailed calculations are given in the article $[Pet\ 94b]$.

## 6.3 Non-Stationary Star and the Trajectory of a Circulating Body

In this sub-chapter a simple model of a non-stationary star is given. The solution contains small time-dependent exterior gravitational effects. The perturbed equations of motion of a test body moving around the non-stationary star are given. The test body moves away from the centre of the star during the epoch of collapsing star and it moves towards the centre during the epoch of expanding star.

The equations of a non-stationary spherically symmetric homogeneous star to Newtonian accuracy as special case of chapter 6.1 (see $[Pet\ 94a]$) are:

$$\frac{\partial v}{\partial t} = -v\frac{\partial v}{\partial r} - \frac{1}{\rho}\frac{\partial pc^2}{\partial r} - \frac{4\pi k}{r^2}\int_0^r x^2\,\rho dx, \tag{6.28a}$$

$$\frac{\partial \rho}{\partial t} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho v) \tag{6.28b}$$

$$\frac{\partial \Pi}{\partial t} = -v\frac{\partial \Pi}{\partial r} - \frac{pc^2}{\rho}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v). \tag{6.28c}$$

The equation of state for a non-relativistic degenerate Fermi gas is

$$\frac{pc^2}{\rho} = \frac{2}{3}\Pi. \tag{6.29}$$

Furthermore, it is assumed that the star is homogeneous, i.e.

$$\rho = \tilde{\rho}(t). \tag{6.30}$$

We use the ansatz

$$v(r,t) = \frac{r}{R_0}\tilde{v}(t), \tag{6.31a}$$

$$\Pi(r,t) = \frac{R^2(t)-r^2}{R_0^2}\widetilde{\Pi}(t) \tag{6.31b}$$

where $R(t)$ denotes the radius of the starand $R_0$ is a fixed arbitrary constant. The gravitational mass to Newtonian accuracy is

$$M_g = 4\pi\int_0^{R(t)} x^2\,\tilde{\rho}dx = \frac{4\pi}{3}\tilde{\rho}R^3. \tag{6.32}$$

It follows by the use of (6.29) to (6.32)

$$\tilde{v} = \frac{R_0}{R(t)}\frac{dR(t)}{dt}, \tag{6.33}$$

$$\widetilde{\Pi} = \beta c^2\left(\frac{R_0}{R(t)}\right)^4 \tag{6.34}$$

where $\beta$ is a constant of integration. Furthermore, the following differential equation $[Pet\ 95a]$ is received

$$\frac{d^2R(t)}{dt^2} = \frac{4}{3}\beta c^2 R_0^2\frac{1}{R(t)^3} - \frac{kM_g}{R(t)^2}. \tag{6.35}$$

This differential equation can be integrated yielding

$$\left(\frac{dR}{dt}\right)^2 = C - \frac{4}{3}\beta c^2\left(\frac{R_0}{R}\right)^2 + 2\frac{kM_g}{R} \tag{6.36}$$

where $C$ is an constant of integration. Knowing a solution $R(t)$ of (6.36) the relations for $\tilde{v}$, $\widetilde{\Pi}$ and $\tilde{\rho}$ are obtained by (6.33), (6.34) and (6.32).

There are two different kinds of solutions:

(1) $C \geq 0$: The radius $R(t)$ contracts to a positive minimum and then it expands for all times.

(2) $C < 0$: The radius $R(t)$ of the star oscillates between a mimimum radius $R_1$ and a maximum radius $R_2$. They are given by

$$R_1 = \left( kM_g - \left( \left( kM_g \right)^2 - \frac{4}{3} \beta c^2 R_0^2 |C| \right)^{1/2} \right) \Big/ |C|$$
$$R_1 = \left( kM_g + \left( \left( kM_g \right)^2 - \frac{4}{3} \beta c^2 R_0^2 |C| \right)^{1/2} \right) \Big/ |C| \tag{6.37}$$

The relations (6.37) give

$$\frac{1}{2}(R_1 + R_2) = \frac{kM_g}{|C|}. \tag{6.38}$$

Equation (6.38) fixes $|C|$ by the mass and the maximum and minimum radius of the star.

The approximate solution of (6.36) has the form

$$R(t) \approx \frac{1}{2}(R_1 + R_2) - \frac{1}{2}(R_2 - R_1) cos \left( sqrt \left( \frac{kM_g}{((R_1 + R_2)/2)^3} \right) t \right). \tag{6.39}$$

Hence, the solution (6.39) describes to Newtonian accuracy a non-singular spherically symmetric, homogeneous, pulsating star.

The period of the oscillation is

$$t_p = 2\pi \sqrt{R_m^3 / kM_g} \tag{6.40a}$$

where

$$R_m = \frac{1}{2}(R_1 + R_2) \tag{6.40b}$$

is the mean radius of the oscillating object. Formula gives for the Sun with

$$k \approx 6.67 \cdot 10^{-8} [cm/(gsec^2)],\ M_\odot \approx 2 \cdot 10^{33} [g],\ R_m \approx 6.96 \cdot 10^{10} [cm]$$

the period of oscillation

$$t_p \approx 9.98 \cdot 10^3 [sec] \approx 166 [min]. \tag{6.41}$$

This result is in good agreement with the experimentally measured value of 160 $[min]$.

The special case $R_1 = R_2$ implies by the use of (6.7) the relation

$$\left(kM_g\right)^2 = \frac{4}{3}\beta c^2 R_0^2 |C|.$$

Then, we get with $\tilde{R} = (R_1 + R_2)/2$ by the use of (6.38)

$$|C| = \frac{kM_g}{\tilde{R}}.$$

Hence, the acceleration (6.35) and the velocity (6.36) at $R(t) = \tilde{R}$ are zero, i.e., we have a stationary star with radius $\tilde{R}$. This result also follows by the use of (6.39). The last two relations give

$$\frac{kM_g}{\tilde{R}} = \frac{4}{3}\beta c^2 \left(\frac{R_0}{\tilde{R}}\right)^2.$$

Relation (6.34) implies for $R(t) = \tilde{R}$

$$\tilde{\Pi} = \frac{3}{4}\frac{kM_g}{\tilde{R}}\left(\frac{R_0}{\tilde{R}}\right)^2.$$

At the centre of the star we get

$$\Pi = \left(\frac{\tilde{R}}{R_0}\right)^2 \tilde{\Pi} = \frac{3}{4}\frac{kM_g}{\tilde{R}}.$$

Hence, we have at the centre of the star by the use of (6.29)

$$\frac{p}{\rho} = \frac{1}{2}\frac{kM_g}{c^2\tilde{R}}.$$

Therefore, we receive a non-singular, spherically symmetric, stationary star where the pressure is given by the above relation.

We will now give the exterior gravitational field of a spherically, non-stationary star to 2-post-Newtonian approximation. The potentials in spherical coordinates are given by (3.4b). We get by (6.23) up to $O\left(\frac{1}{r}\right)$

$$f \approx g \simeq 1 - 2\frac{kM_g}{c^2 r},$$

$$h \approx 1 + 2\frac{kM_g}{c^2 r} + \frac{1}{c^6}\frac{4\pi k}{r}\frac{\partial \tilde{h}}{\partial t},$$

$$\tilde{h} = -\frac{8}{15}\int_0^R x^3 \rho\left(4\frac{pc^2}{\rho}v + v^3 + x\frac{pc^2}{\rho}\frac{\partial v}{\partial x}\right)dx \tag{6.41a}$$

$$+4\pi k\frac{64}{9}\int_0^R x\rho\int_0^x y^3 \rho v dy dx$$

$$+4\pi k\left(\frac{32}{9}\int_0^R x^4 \rho\int_R^x \rho v dy dx + \frac{176}{15}\int_0^R x^2 \rho v\int_0^x y^2 \rho dy dx\right)$$

Elementary calculations yield by the use of (6.30), (6.31), (6.33), (6.34), (6.32) and (6.36) the approximate value

$$\tilde{h} = \frac{8}{35}\frac{M_g}{4\pi}\left(24\frac{kM_g}{R(t)} - C\right)\frac{dR(t)}{dt}. \tag{6.41b}$$

We will now give the motion of a test particle in this gravitational field. The differential equations (2.53) imply by the use of (6.41) for the perturbed orbit $r_0 + \Delta r$ around a circle with radius $r_0$ after some longer calculations (see $[Pet\ 95a]$) the equations

$$\frac{d^2\Delta r}{dt^2} = -\frac{kM_g}{r_0^3}\Delta r - \frac{1}{c^4}\frac{2\pi k}{r_0}\frac{\partial \tilde{h}}{\partial t}. \tag{6.42}$$

This differential equation can be solved by standard methods. We get by suitable initial conditions and elementary longer calculations the perturbed radius

$$\Delta r(t) = -\frac{20}{7}\left(\frac{kM_g}{c^2 r_0}\right)^2 (R(t) - R_1) \tag{6.43a}$$

and the perturbed radial velocity

$$\frac{d}{dt}\Delta r = -\frac{20}{7}\left(\frac{kM_g}{c^2 r_0}\right)^2 \frac{R_2 - R_1}{R_2 + R_1}\left(\frac{2kM_g}{R_1 + R_2}\right)^{1/2} sin\left(\frac{2\sqrt{2kM_g}}{\sqrt[3]{R_1 + R_2}}t\right). \tag{6.43b}$$

The derivation of the perturbed solution is given in $[Pet\ 95a]$ where a factor in the denominator is missing.

Hence, the deviations of the orbit and its velocity from a circle are very small. But this result although very small differs from the corresponding results of general relativity where by the theorem of Birkhoff no change of the orbit arises.

All these results are contained in the articles [$Pet$ 95$a$] and [$Pet$ 10$a$].

## 6.4 Gravitational Radiation from a Binary System

In this sub-chapter 1-post-Newtonian approximations are used to derive the gravitational radiation of a system of objects at large distances from one another. A more explicit formula is given for a binary system. It agrees with the result of general relativity.

We use the 1-post-Newtonian approximation of the potentials (5.8) and the tensor of matter

$$\begin{aligned} T(M)^i_j &= \rho\left(1+\frac{\Pi}{c^2}+\frac{6}{c^2}U+\frac{p}{\rho}+\left(\frac{v}{c}\right)^2\right)v^i v^j \\ &\quad + pc^2\left(1+\frac{2}{c^2}U\right)\delta^i_j - \frac{4}{c^2}\rho V_j v^i, (i; j=1,2,3) \\ &= \rho c v^j\left(1+\frac{\Pi}{c^2}+\frac{6}{c^2}U+\frac{p}{\rho}+\left(\frac{v}{c}\right)^2\right) - \frac{4}{c}\rho V_j, (i=4; j=1,2,3) \\ &= -\rho c v^i\left(1+\frac{\Pi}{c^2}+\frac{2}{c^2}U+\frac{p}{\rho}+\left(\frac{v}{c}\right)^2\right), (i=1,2,3; j=4) \\ &= -\rho c^2\left(1+\frac{\Pi}{c^2}+\frac{2}{c^2}U+\left(\frac{v}{c}\right)^2\right). (i=j=4) \end{aligned} \tag{6.44}$$

Here, the potentials $U$ and $V_i$ are stated by (5.2b) and (5.11). Subsequently, we use the tensors (1.32), the field equations (1.34) and the tensors of the gravitational energy (1.35) and of matter (1.37). It follows from (1.34) by multiplication with $f^{ki}$

$$\left(f^{kl}f^{ij}_{/l}\right)_{/k} = f^{kl}f_{mn}f^{im}_{/k}f^{jn}_{/l} + 4\kappa f^{ik}T^j_k \tag{6.45}$$

Put

$$f^{ij} = \eta^{ij} + \phi^{ij} \tag{6.46}$$

then we get

$$\eta^{kl}\phi^{ij}_{/kl} = \tau^{ij} \tag{6.47a}$$

with

$$\tau^{ij} = -\left(\phi^{kl}\phi^{ij}_{/l}\right)_{/k} + f^{kl} f_{mn}\phi^{im}_{/k}\phi^{jn}_{/l} + 4\kappa f^{ik} T^{j}_{k}. \tag{6.47b}$$

In the following we use the pseudo-Euclidean geometry (1.1) and (1.5). Then, the differential equation (6.47) has the familiar form of a wave equation. The solution for out-going waves is

$$\phi^{ij} = -\frac{1}{4\pi}\int \tau^{ij}\left(x', t - \frac{|x-x'|}{c}\right)/|x - x'|\,dx' \tag{6.48}$$

where the integration is taken over the whole space $R^3$.

Longer calculations are given in the article of Petry [$Pet$ 93$a$]. They follow along the lines of the papers [$Eps$ 75], [$Wag$ 76] and [$Wil$ 77] in studying gravitational radiation by the use of general relativity. The resulting radiation energy $E$ per unit time is given to $O\left(\frac{1}{c^5}\right)$ by

$$\frac{dE}{dt} = -\frac{k}{15c^5}\left\{3\frac{\partial}{\partial t}\left(\frac{\partial^2 I^{ij}}{dct^2}\right)\frac{\partial}{\partial t}\left(\frac{\partial^2 I^{ij}}{\partial ct^2}\right) - \left(\frac{\partial}{\partial t}\left(\frac{\partial^2 I^{ii}}{\partial ct^2}\right)\right)^2\right\} \tag{6.49}$$

where, $I^{ij}$ are the quadrupole moments.

It holds for several point masses $m_A$with velocities $v_a = (v_A^1, v_A^2, v_A^3)$:

$$\left(\frac{\partial^2 I^{ij}}{\partial ct^2}\right) = \sum_A m_A\left(2v_A^i v_A^j + \frac{dv_A^i}{dt}x_A^j + x_A^i\frac{dv_A^j}{dt}\right). \tag{6.50}$$

The application of (6.49) and (6.50) to a binary system gives the gravitational radiation

$$\frac{dE}{dt} = -\frac{8}{15}\frac{k^3\mu^2 m^2}{c^5 r^4}\left(12|v|^2 - 11\left(\frac{dr}{dt}\right)^2\right) \tag{6.51}$$

with the following abbreviations for the two objects $A$ and$B$:

$$m = m_A + m_B,\ \mu = \frac{m_A m_B}{m},\ r = |x_A - x_B|, v = v_A - v_B. \tag{6.52}$$

This result is identical with that of the general relativity theory of Einstein to this accuracy (see [$Wag$ 76] and [$Wil$ 77]. Therefore, the results of both theories agree in the magnitude of the gravitational energy emitted by the binary pulsar system PSR 1913+16 (see Taylor [$Tay$ 79]).

All these results can be found for flat space-time theory of gravitation in $[Pet\ 93a]$ and for the theory of general relativity in the papers $[Eps\ 75]$, $[Wag\ 76]$ and $[Wil\ 77]$.